

Homogeneous spaces adapted to singular integral operators involving rotations

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Abstract

Calderón-Zygmund decompositions of functions have been used to prove weak-type (1,1) boundedness of singular integral operators. In many examples, the decomposition is done with respect to a family of balls that corresponds to some family of dilations. We study singular integral operators T that require more particular families of balls, providing new spaces of homogeneous type. Rotations play a decisive role in the construction of these balls. Boundedness of T can then be shown via Calderón-Zygmund decompositions with respect to these spaces of homogeneous type. We prove weak-type (1,1) and L^p estimates for operators T acting on $L^p(G)$, where G is a homogeneous Lie group. Our results apply to the setting where the underlying group is the Heisenberg group and the rotations are symplectic automorphisms. They also apply to operators that arise from some hydrodynamical problem involving rotations.

1 Introduction and main result

An example where G is abelian. In their paper on a "singular 'winding' integral operator", Farwig, Hishida and Müller [4] considered the following problem. A rigid body rotating with fixed angular velocity is surrounded by an incompressible viscous fluid. This situation can be described by the Navier-Stokes equations and the assumption that the velocity u of the fluid vanishes at infinity and is equal to the local velocity of the body at its surface. While estimating Δu , an integral operator came into play. We consider a similar operator in a setting where $G = \mathbb{R}^2$. Then $d\mu = dx$ is a Haar measure on G . We denote rotations in the plane by $O_t(x) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \cdot x$, where $t \in \mathbb{R}$.

A family of dilations is defined by $D_r(x) = r^{\frac{1}{2}} \cdot x$, where $r > 0$. Let $E(x) = e^{-|x|^2}$ and $E_r(x) = \frac{1}{r} E \circ D_{r^{-1}}(x) = \frac{1}{r} E(\frac{x}{\sqrt{r}})$, $x \in \mathbb{R}^2$. We introduce a linear operator by setting

$$(Tf)(x) = \int_0^\infty (\Delta E_t) * (f \circ O_t)(x) dt,$$

where $\Delta = \partial_1^2 + \partial_2^2$, $f \in \mathcal{S}(\mathbb{R}^2)$ and $(f * g)(x) = \int_{\mathbb{R}^2} f(y)g(x-y) dy$. Setting $\psi = \Delta E$ and $\psi_r = \frac{1}{r} \psi \circ D_{r^{-1}}$, we obtain

$$(Tf)(x) = \int_0^\infty \psi_t * (f \circ O_t)(x) \frac{dt}{t}, \quad f \in \mathcal{S}, x \in \mathbb{R}^2. \quad (1)$$

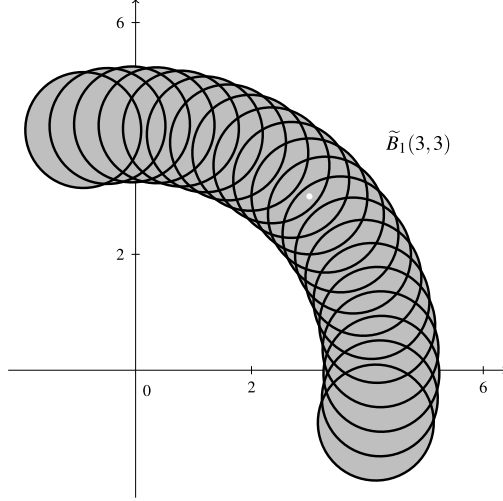


Figure 1: The quasi-ball \tilde{B} of radius 1 and center $(3,3)$.

Note that $\int \psi_r(x) dx = 0$, which implies some cancellation when smooth functions are convoluted with ψ_r and t is small. This cancellations suffices to make the integral in (1) converge. The arguments used in [4] suggest that T can be extended to a bounded linear operator on L^p if $1 < p < \infty$. The operator T is said to be of *weak type (1,1)* or bounded from L^1 to $L^{1,\infty}$, if $\mu(\{x : |Tf(x)| > \lambda\}) \leq C \frac{\|f\|_1}{\lambda}$ for all $\lambda > 0$ and all f . So far, we had no hints on whether T is of weak type $(1,1)$. We will show that this is indeed the case. The proof will rely on quasi-balls \tilde{B} as the one depicted in Figure 1. More precisely, we define the quasi-balls \tilde{B}_r as unions of Euclidean balls of radius $r^{\frac{1}{2}}$ by

$$\tilde{B}_r(z) = \bigcup_{-r \leq s \leq r} \left\{ x \in \mathbb{R}^2 : \|x\|_2 < r^{\frac{1}{2}} \right\} + O_s(z), \quad z \in \mathbb{R}^2, r > 0.$$

An example where G is nonabelian. The set $\mathbb{C}^2 \times \mathbb{R}$ together with the product

$$[(u_1, v_1, s_1), (u_2, v_2, s_2)] = (0, 0, \text{Im}(\overline{u_1}u_2 + \overline{v_1}v_2))$$

is a real Lie algebra. Furthermore, the product

$$x \cdot y = x + y + \frac{1}{2}[x, y], \quad x, y \in \mathbb{C}^2 \times \mathbb{R}$$

defines a real Lie group, the Heisenberg group \mathbb{H}_2 . If $\alpha, \beta \in \mathbb{R}$, the symplectic automorphisms $O_t(u, v, s) = (ue^{i\alpha t}, ve^{i\beta t}, s)$ can be viewed as rotations and the automorphisms $D_t(u, v, s) = (tu, tv, t^2s)$ as nonisotropic dilations, yielding a noncommutative example for the general case.

The more general setting presented below has been described and explained by Folland and Stein [6, Chap. 1]. We add the notion of rotations. For the sake of simplicity, the assumptions are slightly redundant.

Assumptions Let G be a connected and simply connected nilpotent Lie group with Lie algebra \mathfrak{g} , $\dim \mathfrak{g} = n \geq 2$. We identify G with \mathfrak{g} via the group exponential function, so G is the manifold \mathfrak{g} together with a group product that is given by the Campbell-Hausdorff formula and the Lie product $[\cdot, \cdot]$ on \mathfrak{g} . In this setting the neutral element of G equals $0_{\mathfrak{g}}$. We have the identities $x^{-1} = -x$, $xy = x + y + \frac{1}{2}[x, y] + \dots$, $\exp_G = \text{Id}$ and $\text{Aut } G = \text{Aut } \mathfrak{g} \subseteq \text{GL}(\mathfrak{g})$.

We assume that A is a diagonalizable linear operator on \mathfrak{g} whose eigenvalues are positive and that $\{\exp(A \log t)\}_t$ is a family of Lie algebra automorphisms. Furthermore, we assume that $\{D_t\}_{t>0}$ is a family of Lie group automorphisms such that $dD_t(e) = \exp(A \log t)$. Then $\{D_t\}_{t>0}$ is called a family of *dilations* and G is called a *homogeneous group* with respect to $\{D_t\}_t$. $Q = \text{trace } A$ is called its *homogeneous dimension*. The maps $D_t = \exp(A \log t)$ are linear.

Let $O : \mathbb{R} \rightarrow \text{Aut}(G)$ be a continuous homomorphism whose image is relatively compact. We also write $\{O_t\}_{t \in \mathbb{R}}$ instead of O and refer to $\{O_t\}_{t \in \mathbb{R}}$ as the family of *rotations*. We assume that rotations commute with dilations, which amounts to saying that the eigenspaces of nontrivial dilations are left invariant by rotations.

From now on we identify \mathfrak{g} (and hence G) with \mathbb{R}^n . The map $t \mapsto \det O_t$ is a continuous homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R} \setminus \{0\}, \cdot)$. Since $\det O_{\mathbb{R}}$ is a subset of the compact set $\det(O_{\mathbb{R}})$ in \mathbb{R} , it is also bounded. Hence for every t we have $\det O_t = 1$ and $O_t \in \text{SL}(n)$. Since the closure of $\{O_t : t \in \mathbb{R}\}$ is a compact Lie subgroup of the connected group $\text{SL}(n)$, it can be conjugated into the maximal compact subgroup $\text{SO}(n)$ of $\text{SL}(n)$ [8]. Therefore, the identification of \mathfrak{g} with \mathbb{R}^n can be done in such a way that $O_t \in \text{SO}(n)$ for all t , and we will proceed on this assumption. We define $\mathcal{S}(G)$ to be $\mathcal{S}(\mathbb{R}^n)$ and use the Euclidean Schwartz norms

$$\|f\|_{(N)} = \sup_{|\alpha| \leq N, x \in G} \langle x \rangle^N |\partial^\alpha f(x)|, \quad \langle x \rangle = \left(1 + \|x\|_2^2\right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n.$$

Instead, Schwartz norms defined in terms of invariant vector fields and a homogeneous norm could be used, see [6, p. 35].

We use the Lebesgue measure dx on G and denote it by μ when measuring sets, that is, $\mu(M) = \int_M dx$. This is a left- and right-invariant Haar measure. Let L^p and $L^{1,\infty}$ be the Lorentz spaces $L^p(G, dx)$ and $L^{1,\infty}(G, dx)$ respectively. Note that $\|\cdot\|_{1,\infty}$ is a quasinorm, and that $L^{1,\infty}$ is complete [7, Thm. 1.4.11.]. We denote the convolution of f and g by $(f * g)(x) = \int_G f(xy^{-1})g(y)dy$. For any function $\psi : G \rightarrow \mathbb{C}$ and $t > 0$ we denote by $\psi_t = t^{-Q}\psi \circ D_{t^{-1}}$ the L^1 -invariantly dilated function ψ_t .

The singular integral operator We fix some $\psi \in \mathcal{S}(G)$ with $\int \psi(x)dx = 0$ and define

$$(Tf)(x) = \int_0^\infty [\psi_t * (f \circ O_t)](x) \frac{dt}{t}, \quad f \in \mathcal{S}, x \in G. \quad (2)$$

With techniques similar to the ones used in the proof of Lemma 3, it can be shown that (2) converges. For the proof of the following theorem, we will rely on a definition of T as an operator on $L^2(G)$, which is compatible with (2).

Theorem 1. *Let $1 < p \leq 2$ and T as in (2). There exists a constant C such that for all $f \in \mathcal{S}$ and all $\lambda > 0$ we have the estimates*

$$\mu(\{x \in G : |Tf(x)| > \lambda\}) \leq C \frac{\|f\|_1}{\lambda} \quad (3)$$

and

$$\|Tf\|_p \leq C \|f\|_p. \quad (4)$$

Hence T has a unique extension to a bounded linear operator on L^p , $p \in]1, 2]$ and a unique extension to a bounded linear operator from L^1 to $L^{1,\infty}$. On the premise that $\psi(O_t x) = \psi(x)$ for all t and x , the preceding is also true for $2 < p < \infty$.

2 Prerequisites

Homogeneous norms. A continuous function $|\cdot| : G \rightarrow [0, \infty)$ is said to be a homogeneous norm on G with respect to $\{D_t\}_t$ if it satisfies $|x| = 0 \Leftrightarrow x = 0$, $|x^{-1}| = |x|$ and $|D_t x| = t|x|$ for all $t > 0$ and $x \in G$. Some authors require homogeneous norms to be smooth away from the origin. There exists a homogeneous norm $|\cdot|$ that is invariant under rotations; that is,

$$|O_t x| = |x| \quad (5)$$

for all x, t . For a hint on how to produce such homogeneous norms, see the example in Sect. 7. We will keep one such homogeneous norm fixed. The terms $|x^{-1}y|$ and $|xy^{-1}|$ define left-invariant and right-invariant quasi-distance functions respectively. These quasi-distances are symmetric and coincide if G is abelian. We refer to them as the *homogeneous distance* between x and y . We use the term *quasi-metric* for symmetric quasi-distance functions; namely, if d is a quasi-metric then $x = y \Leftrightarrow d(x, y) = 0$, $d(x, y) = d(y, x)$, and there is a constant κ such that $d(x, y) \leq \kappa(d(x, z) + d(z, y))$.

Balls and spheres. Balls and spheres with center 0 and with respect to $|\cdot|$ are defined by $B_r = \{x \in G : |x| < r\}$ and $S_r = \{x \in G : |x| = r\}$ respectively. The ball with center z and with respect to the left- and right-invariant quasi-metrics are equal to $z \cdot B_r$ and $B_r \cdot z$ respectively.

Integration. The definition of Q is natural in the sense that there exists a constant C so that $\int_{B_r \cdot z} dx = Cr^Q$, for all $z \in G$, $r > 0$ and that $\det D_r = r^Q$. Furthermore, there exists a positive Borel measure σ on S_1 such that all $f \in L^1(G)$ can be integrated using *spherical coordinates* by

$$\int_G f(x) dx = \int_0^\infty \int_{S_1} f(D_r x) d\sigma(x) r^{Q-1} dr. \quad (6)$$

Since $O_t \in \text{SL}(n)$, μ is invariant under rotations:

$$\int f(x) dx = \int (f \circ O_t)(x) dx \quad (7)$$

for every f and t . Note that $f_r * g_r = (f * g)_r$, and that $(f * g) \circ O_t = (f \circ O_t) * (g \circ O_t)$, for all $f, g \in L^1(G)$, $t \in \mathbb{R}$, $s > 0$.

Constants. We will use miscellaneous constants $C, C', C_1 > 0$ etc. whose values vary from line to line and who may depend on the geometric setting, for example, on the homogeneous group. Occasionally we write $a(x) \lesssim b(x)$ or $a \lesssim b$ to indicate that there is a constant C such that $a(x) \leq C \cdot b(x)$ for all x . Furthermore, $a \simeq b$ shall mean that for some C_1, C_2 , we have $a(x) \leq C_1 \cdot b(x) \leq C_2 \cdot a(x)$ for all x .

Norm estimates. By γ we denote the smallest eigenvalue of A and by Γ the greatest eigenvalue of A .

For any vector space norm $\|\cdot\|$ and any relatively compact neighborhood U of the origin there exist constants $C_1, C_2 > 0$ such that we have the norm estimate

$$|x|^\Gamma \leq C_1 \|x\| \leq C_2 |x|^\gamma \quad \text{for all } x \in U \quad (8)$$

and

$$|x|^\gamma \leq C_1 \|x\| \leq C_2 |x|^\Gamma \quad \text{for all } x \in G \setminus U. \quad (9)$$

Cancellation. The quantity $|\psi(x) - \psi(y)|$ can be estimated in various ways in terms of the distance between x and y , for example, see [6, p. 28]. The following lemma is fine for our purpose.

Lemma 2. *For any $l \in \mathbb{N}$ there exists a constant C and a Schwartz norm $\|\cdot\|_{(N)}$ such that for all $\psi \in \mathcal{S}(G)$ and all $x, y \in G$ we have*

$$|\psi(x) - \psi(y)| \leq C \|\psi\|_{(N)} \cdot |x^{-1}y|^\gamma \cdot \left(\frac{1}{(1 + \|x\|_2)^l} + \frac{1}{(1 + \|y\|_2)^l} \right).$$

Proof. We find N and C such that the Lemma is true on the additional assumption that $|x^{-1}y| \geq 1$, because $\psi \in \mathcal{S}$. We will possibly increase the values of N and C later. Now let $|x^{-1}y| \leq 1$. Since the map

$$h : G \times G \rightarrow G, \quad (x, z) \mapsto x - xz$$

is smooth, any derivative of h is bounded on any compact set. Furthermore, we have $h(x, 0) = 0$. Now compactness and the norm estimate (8) yield for any x with $|x| \leq 1$

$$\|x - y\|_2 = \|h(x, x^{-1}y)\|_2 \leq C_1 \|x^{-1}y\|_2 \leq C_2 |x^{-1}y|^\gamma. \quad (10)$$

If $t = |x| \geq 1$, using (10) with $x - y$ replaced by $D_{t^{-1}}x - D_{t^{-1}}y$, we obtain

$$\begin{aligned} \|x - y\|_2 &= \|D_t(D_{t^{-1}}x - D_{t^{-1}}y)\|_2 \leq \|D_t\|_{\text{op}} \|D_{t^{-1}}x - D_{t^{-1}}y\|_2 \\ &\leq Ct^\Gamma |(D_{t^{-1}}x)^{-1}(D_{t^{-1}}y)|^\gamma = Ct^{\Gamma-\gamma} |x^{-1}y|^\gamma = C|x|^{\Gamma-\gamma} |x^{-1}y|^\gamma. \end{aligned} \quad (11)$$

Interchanging the roles of x and y in (11) and combining with (10), we get

$$\|x - y\|_2 \leq C(1 + \min\{|x|, |y|\}^{\Gamma-\gamma}) |x^{-1}y|^\gamma.$$

With the help of (9) we find some $p \in \mathbb{N}$ such that $|z|^{\Gamma-\gamma} \leq C\|z\|_2^p$ for all $z \in G \setminus B_1$. Setting $R = \min\{\|x\|_2, \|y\|_2\}$, it follows that

$$\|x - y\|_2 \leq C(1 + R)^p |x^{-1}y|^\gamma. \quad (12)$$

We choose an integration path from x to y in $M = \{z \in G : \|z\|_2 \geq R\}$ such that the length of this path is bounded by a constant multiple of $\|x - y\|_2$. Let us estimate $|\psi(x) - \psi(y)|$ by integrating along that path. Since $\psi \in \mathcal{S}$, we find a number N such that

$$\left\| (\nabla \psi)(1 + \|\cdot\|_2)^{l+p} \right\|_\infty \leq \|\psi\|_{(N)}. \quad (13)$$

Putting (12) and (13) together, we obtain

$$|\psi(x) - \psi(y)| \leq C_1 \|(\nabla \psi)1_M\|_\infty \|x - y\|_2 \leq C_2 \frac{\|\psi\|_{(N)}}{(1+R)^{l+p}} (1+R)^p |x^{-1}y|^\gamma.$$

Finally, because of

$$\frac{1}{(1+R)^l} \leq \frac{1}{(1+\|x\|_2)^l} + \frac{1}{(1+\|y\|_2)^l},$$

we get the desired result. \square

Lemma 3. *There is a constant C and a Schwartz norm $\|\cdot\|_{(N)}$ such that for all $\varphi \in \mathcal{S}(G)$ with $\int \varphi = 0$, all $\psi \in \mathcal{S}$, and all $0 < s < 1$ we have*

$$\|\psi * \varphi_s\|_1 \leq C \|\psi\|_{(N)} \|\varphi\|_{(N)} s^\gamma \quad \text{and} \quad \|\psi_s * \varphi\|_1 \leq C \|\psi\|_{(N)} \|\varphi\|_{(N)} s^\gamma.$$

Proof. We give a proof of the first inequality. Using Lemma 2 with $l = n + 1$, we obtain

$$\begin{aligned} \|\psi * \varphi_s\|_1 &= \int \left| \int \psi(xy^{-1}) \varphi_s(y) dy \right| dx \\ &= \int \left| \int [\psi(xy^{-1}) - \psi(x)] \varphi_s(y) dy \right| dx \\ &= \int \int |\psi(xy^{-1}) - \psi(x)| \cdot |\varphi_s(y)| dy dx \\ &\leq C \|\psi\|_{(N)} \int \int |y|^\gamma \left(\frac{1}{(1+\|x\|_2)^{n+1}} + \frac{1}{(1+\|xy^{-1}\|_2)^{n+1}} \right) |\varphi_s(y)| dy dx \\ &\leq C \|\psi\|_{(N)} \int |y|^\gamma \int \left(\frac{1}{(1+\|x\|_2)^{n+1}} + \frac{1}{(1+\|xy^{-1}\|_2)^{n+1}} \right) dx |\varphi_s(y)| dy \\ &\leq C \|\psi\|_{(N)} \left(\int \frac{1}{(1+\|x\|_2)^{n+1}} dx \right) \int |y|^\gamma |\varphi_s(y)| dy \\ &\leq C \|\psi\|_{(N)} \int |D_s y|^\gamma \varphi(y) dy \\ &\leq C \|\psi\|_{(N)} s^\gamma \int |y|^\gamma \varphi(y) dy \\ &\leq C \|\psi\|_{(N)} \|\varphi\|_{(N)} s^\gamma. \end{aligned}$$

\square

3 L^2 results

The space $L^2(G)$ together with the product $\langle f|g \rangle = \int_G f(x) \overline{g(x)} dx$ is a Hilbert space. This allows to extend the linear operator T defined by (2) to a bounded operator on $L^2(G)$, yielding (4) for $p = 2$. Observe that the operators

$$A_t : L^2(G) \rightarrow L^2(G), \quad A_t f = \psi_t * (f \circ O_t), \quad t > 0.$$

are bounded by Young's inequality, which is valid in the context of homogeneous groups, and by (7). Namely, we have $\|A_t\| \leq \|\psi\|_1$ for every $t > 0$; and $A_t^* f = (\psi_t^* * f) \circ O_{-t}$, where $\psi_t^*(x) = \overline{\psi(x^{-1})}$. The set $\mathcal{E} = \{E \subseteq]0, \infty[: E \text{ measurable and } \int_E \frac{dt}{t} < \infty\}$ ordered by inclusion is a directed set.

Theorem 4. *The net $(\int_E A_t \frac{dt}{t})_{E \in \mathcal{E}}$ converges in the weak operator topology to a bounded linear operator \tilde{T} on $L^2(G)$, whose restriction to \mathcal{S} equals T .*

Proof. The main task is to show that there is a function h such that

$$\|A_t A_s^*\|^{\frac{1}{2}} \leq h(t, s) \quad \text{and} \quad \|A_t^* A_s\|^{\frac{1}{2}} \leq h(t, s) \quad (14)$$

and

$$\sup_{s>0} \int_0^\infty h(t, s) \frac{dt}{t} < \infty. \quad (15)$$

Then the proof will be finished by using a continuous version of Cotlar's lemma [5, Appendix B]. Let $s, t > 0$ and $f \in L^2(G)$. We have the estimates

$$\begin{aligned} \|A_t A_s^* f\|_2 &= \|\psi_t * [(\psi_s^* * f) \circ O_s^{-1} \circ O_t]\|_2 \\ &= \|(\psi_t \circ O_{s-t}) * (\psi_s^* * f)\|_2 \\ &= \|[(\psi \circ O_{s-t})_t * \psi_s^*] * f\|_2 \\ &\leq \|(\psi \circ O_{s-t})_t * \psi_s^*\|_1 \|f\|_2, \end{aligned}$$

$$\begin{aligned} \|A_t^* A_s f\|_2 &= \|(\psi_t^* * [\psi_s * (f \circ O_s)]) \circ O_t^{-1}\|_2 \\ &\leq \|\psi_t^* * \psi_s\|_1 \|f\|_2. \end{aligned}$$

Setting

$$h(t, s) = \|(\psi \circ O_{s-t})_t * \psi_s^*\|_1^{\frac{1}{2}} + \|\psi_t^* * \psi_s\|_1^{\frac{1}{2}},$$

we obtain (14). It remains to show (15). This can be done by using the Schwartz norms of $\psi \circ O_{s-t}$, which are bounded uniformly in s and t , and Lemma 3. Let us consider an arbitrary family $\{\varphi^{s,t}\}_{s,t>0}$ of Schwartz functions such that $\|\varphi^{s,t}\|_{(N)} \leq C_N$ for all $s, t > 0$. Furthermore, assume $\chi \in \mathcal{S}$ and $\int \varphi^{s,t} = \int \chi = 0$ for all $s, t > 0$. Then there

exists a Schwartz norm $\|\cdot\|_{(N)}$ such that for all $s > 0$ the estimate

$$\begin{aligned}
\int_0^\infty \|\varphi_t^{s,t} * \chi_s\|_1^{\frac{1}{2}} \frac{dt}{t} &= \int_0^\infty \|\varphi_{ts}^{s,ts} * \chi_s\|_1^{\frac{1}{2}} \frac{dt}{t} \\
&= \int_0^\infty \|(\varphi_t^{s,ts} * \chi)_s\|_1^{\frac{1}{2}} \frac{dt}{t} \\
&= \int_0^\infty \|\varphi_t^{s,ts} * \chi\|_1^{\frac{1}{2}} \frac{dt}{t} \\
&= \int_0^1 \|\varphi_t^{s,ts} * \chi\|_1^{\frac{1}{2}} \frac{dt}{t} + \int_0^1 \|\varphi_{t^{-1}}^{s,t^{-1}s} * \chi\|_1^{\frac{1}{2}} \frac{dt}{t} \\
&= \int_0^1 \|\varphi_t^{s,ts} * \chi\|_1^{\frac{1}{2}} \frac{dt}{t} + \int_0^1 \|\varphi^{s,t^{-1}s} * \chi_t\|_1^{\frac{1}{2}} \frac{dt}{t} \\
&\leq C \left(\sup_{s,t} \|\varphi^{s,t}\|_{(N)}^{\frac{1}{2}} \right) \|\chi\|_{(N)}^{\frac{1}{2}} \int_0^1 t^{\frac{\gamma}{2}} \frac{dt}{t} \\
&\leq C \left(\sup_{s,t} \|\varphi^{s,t}\|_{(N)}^{\frac{1}{2}} \right) \|\chi\|_{(N)}^{\frac{1}{2}}
\end{aligned}$$

holds. Setting $\varphi^{s,t} = \psi \circ O_{s-t}$, $\chi = \psi^*$; and $\varphi^{s,t} = \psi^*$, $\chi = \psi$; yields (15). It follows that $\tilde{T} = \lim_{E \in \mathcal{C}} \int_E A_t \frac{dt}{t}$ for some bounded operator \tilde{T} on $L^2(G)$.

Let f and g be Schwartz functions. We have $\|A_t f\|_\infty \leq \|\psi_t\|_\infty \|f\|_1 \leq Ct^{-Q}$ and furthermore, for $0 < t < 1$, Lemma 3 yields $\|A_t f\|_1 \leq Ct^\gamma \|\psi\|_{(N)} \|f\|_{(N)}$. This results in

$$\begin{aligned}
\int_0^1 \int_G |t^{-1} A_t(x) \cdot \overline{g(x)}| dx dt &< \int_0^1 t^{-1} \|A_t f\|_1 \cdot \|g\|_\infty dt < \infty, \\
\int_1^\infty \int_G |t^{-1} A_t(x) \cdot \overline{g(x)}| dx dt &< \int_1^\infty t^{-1} \|A_t f\|_\infty \cdot \|g\|_1 dt < \infty,
\end{aligned}$$

that is, $(t, x) \mapsto |t^{-1} A_t(x) \cdot \overline{g(x)}|$ is integrable. Fubini's Theorem yields

$$\begin{aligned}
&\langle \tilde{T} f | g \rangle \\
&= \lim_E \int_E \langle A_t f | g \rangle \frac{dt}{t} \\
&= \int_0^\infty \int_G t^{-1} A_t(x) \cdot \overline{g(x)} dx dt \\
&= \int_G \int_0^\infty t^{-1} A_t(x) dt \cdot \overline{g(x)} dx \\
&= \langle T f | g \rangle,
\end{aligned} \tag{16}$$

showing $\tilde{T}|_{\mathcal{S}} = T$. □

From now on we denote \tilde{T} by T .

4 A space of homogeneous type

We now define *quasi-balls* $\tilde{B} \subseteq G$. For any $r > 0$ and $y \in G$ we set

$$\tilde{B}_r(y) = \bigcup_{s \in [-r, r]} B_r \cdot (O_s y).$$

Then by (5) we have $x \in \tilde{B}_r(y) \Leftrightarrow y \in \tilde{B}_r(x) \Leftrightarrow \exists s : |s| \leq r \wedge |x \cdot O_s y^{-1}| < r$ and $O_s(\tilde{B}_r(y)) = \tilde{B}_r(O_s y)$. We show that the balls \tilde{B} possess the engulfing and doubling properties as described by Stein in [10, p. 8].

Theorem 5. *There exist constants $C, k > 0$ such that for all $x, y \in G$ and $t > 0$*

$$\tilde{B}_t(y) \cap \tilde{B}_t(x) \neq \emptyset \Rightarrow \tilde{B}_t(y) \subseteq \tilde{B}_{kt}(x), \quad (17)$$

$$\mu(\tilde{B}_{2t}(x)) \leq C \cdot \mu(\tilde{B}_t(x)). \quad (18)$$

Proof. Choose $a_1, \dots, a_l \in G$ such that $B_2 \subseteq \bigcup_{k=1}^l a_k B_1$. These exist, since B_2 is relatively compact and $\{z \cdot B_1\}_{z \in G}$ is an open covering of \bar{B}_2 . It follows that $B_{2t} = D_t B_2 \subseteq D_t(\bigcup_{k=1}^l a_k B_1) \subseteq \bigcup_{k=1}^l (D_t a_k)(D_t B_1) = \bigcup_{k=1}^l (D_t a_k) B_t$. Finally we obtain

$$\begin{aligned} \mu(\tilde{B}_{2t}(x)) &= \mu(B_{2t} \cdot \{O_s x : s \in [-2t, 2t]\}) \\ &\leq \mu\left(\left(\bigcup_{k=1}^l (D_t a_k) \cdot B_t\right) \cdot \left(\bigcup_{\sigma=\pm t} \{O_{s+\sigma} x : |s| \leq t\}\right)\right) \\ &\leq \mu\left(\bigcup_{k=1}^l \bigcup_{\sigma=\pm t} (D_t a_k) \cdot B_t \cdot \{O_{s+\sigma} x : |s| \leq t\}\right) \\ &\leq \sum_{k=1}^l \sum_{\sigma=\pm t} \mu((D_t a_k) \cdot B_t \cdot \{O_{s+\sigma} x : |s| \leq t\}) \\ &\leq \sum_{k=1}^l \sum_{\sigma=\pm t} \mu(O_{-\sigma}(B_t \cdot \{O_{s+\sigma} x : |s| \leq t\})) \\ &\leq \sum_{k=1}^l \sum_{\sigma=\pm t} \mu(B_t \cdot \{O_s x : |s| \leq t\}) \\ &= 2l\mu(\tilde{B}_t(x)). \end{aligned}$$

This is the doubling property (18). The engulfing property (17) is known to be true when the quasi-balls \tilde{B}_t are replaced by the simpler quasi-balls B_t . So we may choose a constant $k \geq 3$ such that (17) holds with B_t instead of \tilde{B} , and proceed to prove (17).

Now assume that $\tilde{B}_t(x) \cap \tilde{B}_t(y) \neq \emptyset$. Choose $a, b \in [-t, t]$ such that $B_t \cdot (O_a x) \cap B_t \cdot (O_b y) \neq \emptyset$. Property (17) with B instead of \tilde{B} yields $B_t(O_a x) \subseteq B_{kt}(O_b y)$. Let $s \in [-t, t]$. Rotating both sets with O_{s-a} and keeping in mind that $|s - a + b| \leq 3t \leq kt$, we conclude that $B_t \cdot (O_s x) \subseteq B_{kt}(O_{s-a+b} y) \subseteq \tilde{B}_{kt}(y)$, that is, $\tilde{B}_t(x) \subseteq \tilde{B}_{kt}(y)$. \square

Corollary 6. *The Hardy-Littlewood maximal operator*

$$(Mf)(x) = \sup_{r>0} \frac{1}{\mu(\tilde{B}_r(x))} \int_{\tilde{B}_r(x)} |f(y)| dy$$

is of weak type $(1,1)$.

For a proof, see [10] for example. The quasi-balls \tilde{B} define a quasi-metric

$$d(x,y) = \inf \{ r > 0 : x \in \tilde{B}_r(y) \}. \quad (19)$$

This quasi-metric yields a space of homogeneous type in the sense of Coifman and Weiss [2]. Note that for any quasi-metric there are constants $C_1, C_2, C_3 \geq 1$ such that for all x, y, \bar{y} we have

$$C_1 \cdot d(y, \bar{y}) < d(x, y) \Rightarrow d(x, y) \leq C_2 d(x, \bar{y}) \leq C_3 d(x, y). \quad (20)$$

5 The integral kernel

In this section we study singular integral kernels $K(x,y)$ related to T . Let $\eta : G \rightarrow \mathbb{C}$ be a continuous function such that $\|\eta\|_\infty = C_1 < \infty$ and $\sup_x |\eta(x)|x|^N| = C_2 < \infty$ for some $N > Q$.

Lemma 7. *The integral*

$$K_\eta(x,y) = \int_0^\infty \eta_t(x \cdot O_{-t}y^{-1}) \frac{dt}{t}, \quad x \neq y, \quad x, y \in G \quad (21)$$

converges and defines a continuous function $K : G \times G \setminus \{(x,x) : x \in G\} \rightarrow \mathbb{C}$. There exists a constant C such that

$$|K_\eta(x,y)| \leq C \cdot [d(x,y)]^{-Q} \quad (22)$$

for all $x, y \in G$, $x \neq y$. The estimate (22) remains true if in (21) O_{-t} is replaced by O_t or $x \cdot O_{-t}y^{-1}$ by $(O_{-t}x) \cdot y^{-1}$.

Proof. Assume that $x_0, y_0 \in G$ and that $d(x_0, y_0) = \varepsilon > 0$. Then for any $0 \leq t \leq \varepsilon$ we have

$$|x_0 \cdot O_{-t}y_0^{-1}| \geq \varepsilon$$

and there is a $\delta > 0$ such that

$$|x \cdot O_{-t}y^{-1}| \geq \frac{\varepsilon}{2} \quad \text{for all } x \in B_\delta \cdot x_0, \quad y \in B_\delta \cdot y_0 \quad \text{and} \quad 0 \leq t \leq \varepsilon. \quad (23)$$

We will construct a function $H : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$|\eta_t(x \cdot O_{-t}y^{-1})| \leq H(t) \quad \text{for all } x \in B_\delta \cdot x_0, \quad y \in B_\delta \cdot y_0 \quad \text{and} \quad 0 \leq t \leq \varepsilon \quad (24)$$

and such that

$$\int_0^\infty H(t) \frac{dt}{t} \leq C\varepsilon^{-Q}, \quad (25)$$

where C is a constant depending on η , but not on x or y . Then the estimate (22) is obvious. Furthermore, the convergence of the integral and the continuity of K follow by the dominated convergence theorem. To prove (25), note that $\eta_t(x \cdot O_{-t}y^{-1}) =$

$t^{-Q}\eta \circ h(t)$, where $h(t) = D_{t^{-1}}(x \cdot O_{-t}y^{-1})$. Because of (23), $|h|$ is bounded from below by $|h(t)| = t^{-1}|x \cdot O_{-t}y^{-1}| \geq \frac{1}{2}t^{-1}\varepsilon \geq \frac{1}{2}$ for $0 \leq t \leq \varepsilon$, and η is bounded from above by

$$|\eta \circ h(t)| \leq \frac{C_2}{|h(t)|^N} \leq C_2 2^N \cdot t^N \cdot \varepsilon^{-N}. \quad (26)$$

Then $H(t) = t^{-Q} \begin{cases} C_1 & t > \varepsilon \\ C_2 2^N t^N \varepsilon^{-N} & t \leq \varepsilon \end{cases}$ yields (24) and (25), which finishes the proof of Lemma 7. \square

For example, $\eta = \psi$ and $\eta = |\psi|$ satisfy the assumptions of Lemma 7. Note that (22) is weaker than the condition

$$|K(x, y)| \leq \frac{C}{\mu(\tilde{B}_{d(x, y)}(y))}$$

required for standard Calderón-Zygmund kernels, see Sect. 8.

The pointwise estimate (22) does not suggest integrability of $K(\cdot, y)$:

$$\int_G [d(x, y)]^{-Q} dx \geq \int_G |xy^{-1}|^{-Q} dx = C \int_0^\infty r^{-Q} r^{Q-1} dr = \infty.$$

Nonetheless, if $K_\eta = {}^\varepsilon K_\eta + K_\eta^\varepsilon$, where

$${}^\varepsilon K_\eta(x, y) = \int_0^\varepsilon \eta_t(x \cdot O_{-t}y^{-1}) \frac{dt}{t}, \quad x \neq y, x, y \in G, \quad (27)$$

$$K_\eta^\varepsilon(x, y) = \int_\varepsilon^\infty \eta_t(x \cdot O_{-t}y^{-1}) \frac{dt}{t}, \quad x, y \in G, \quad (28)$$

then ${}^\varepsilon K_\eta(\cdot, y_0)$ is a function that is integrable at infinity and K_η^ε is a bounded function such that for every y_0 the kernel satisfies some estimate $K_\eta^\varepsilon(x, y_0) \lesssim |x|^{-Q}$ as x tends to infinity. If $\eta \geq 0$ and $\eta(0) > 0$, then even $K_\eta^\varepsilon(x, y_0) \gtrsim |x|^{-Q}$.

Lemma 8. *There is a constant C such that for all y , η , and ε , we have*

$$\int_{d(x, y) > \varepsilon} |{}^\varepsilon K_\eta(x, y)| dx \leq C \cdot \sup_x |\eta(x)| |x|^N.$$

Proof. Without loss of generality we assume $\eta \geq 0$ and C_2 as before. Assume that $0 < \varepsilon \leq d(x, y)$ and $0 < t < \varepsilon$. Then we have $|x \cdot O_{-t}y^{-1}| \geq \varepsilon$. Substituting z for

$x \cdot O_{-t}y^{-1}$, we obtain $|z| \geq \varepsilon$ and, using spherical coordinates (6), we obtain the estimate

$$\begin{aligned}
& \int_{d(x,y) > \varepsilon} \int_0^\varepsilon \eta_t(x \cdot O_{-t}y^{-1}) \frac{dt}{t} dx \\
& \leq \int_0^\varepsilon \int_{|z| \geq \varepsilon} \eta_t(z) dz \frac{dt}{t} \\
& \leq \int_0^\varepsilon \int_{|z| \geq \frac{\varepsilon}{t}} \eta(z) dz \frac{dt}{t} \\
& \leq \sigma(S_1) \int_0^\varepsilon \int_{\frac{\varepsilon}{t}}^\infty \frac{C_2}{r^N} r^{Q-1} dr \frac{dt}{t} \\
& \leq C \cdot C_2 \int_0^\varepsilon \left(\frac{\varepsilon}{t}\right)^{Q-N} \frac{dt}{t} \\
& \leq C \cdot C_2.
\end{aligned}$$

□

Lemma 9. *The operator T is expressible as a singular integral as follows. For all $f \in L^2(G)$ with compact support and all $x \in G \setminus \text{supp } f$, the integrals*

$$Tf(x) = \int_G K_\psi(x, y) f(y) dy \quad \text{and} \quad T^*f(x) = \int_G \overline{K_\psi(y, x)} f(y) dy$$

converge and equality holds for almost all $x \in G \setminus \text{supp } f$.

Proof. The function $K_{|\psi|}$ is continuous by Lemma 7. If $f, g \in L^2(G)$ have compact support and $\text{supp } f \cap \text{supp } g = \emptyset$, then we have the estimate

$$\begin{aligned}
\infty & > \int_G \int_G K_{|\psi|}(x, y) |f(y)| |g(x)| dy dx \\
& = \int_G \int_G \int_0^\infty \left| \psi_t(x \cdot O_{-t}y^{-1}) \cdot f(y) \cdot \overline{g(x)} \right| \frac{dt}{t} dx dy.
\end{aligned}$$

Tonelli's and Fubini's Theorems imply that

$$\begin{aligned}
& \int_G \int_G K_\psi(x, y) f(y) dy \overline{g(x)} dx \\
& = \int_G \int_G \int_0^\infty \psi_t(x \cdot O_{-t}y^{-1}) \cdot f(y) \cdot \overline{g(x)} \frac{dt}{t} dx dy \\
& = \int_0^\infty \int_G \int_G \psi_t(x \cdot O_{-t}y^{-1}) \cdot f(y) dy \overline{g(x)} dx \frac{dt}{t} \\
& = \int_0^\infty \int_G \int_G \psi_t(x \cdot y^{-1}) \cdot f(O_t y) dy \overline{g(x)} dx \frac{dt}{t} \\
& = \int_0^\infty \langle \psi_t * (f \circ O_t) | g \rangle \frac{dt}{t} \\
& = \int_0^\infty \langle A_t f | g \rangle \frac{dt}{t} \\
& = \langle Tf | g \rangle,
\end{aligned}$$

where $O_t y$ has been substituted for y . A similar calculation yields

$$\int_G \int_G \overline{K_\psi(y, x)} f(y) dy \overline{g(x)} dx = \langle T^* f | g \rangle.$$

Finally, let $\overline{B} \subseteq G$ be a compact ball with rational radius and rational center such that $\overline{B} \subseteq G \setminus \text{supp } f$. Then $\int_G K_\psi(x, y) f(y) dy$ converges for all $x \in \overline{B}$,

$$\langle T f | g \rangle = \left\langle \int_G K_\psi(\cdot, y) f(y) dy \middle| g \right\rangle$$

holds for all $g \in L^2(B)$ and hence $T f(x) = \int_G K_\psi(x, y) f(y) dy$ for almost all $x \in \overline{B}$. Since there are only countably many balls \overline{B} and every $x \in G \setminus \text{supp } f$ is contained in such a ball, we have equality for almost all $x \in G \setminus \text{supp } f$.

Similar arguments apply to T^* . \square

We will need the following technical lemma.

Lemma 10. *There is a constant $C_1 > 0$ such that, for all $x \in G$, $t > 0$ and $s \in [-\frac{t}{2}, \frac{t}{2}]$,*

$$|(D_{t-s}^{-1} x)^{-1} (D_t^{-1} x)| \leq C_1 |D_t^{-1} x| \cdot \left(\frac{|s|}{t} \right)^{\frac{1}{r}}.$$

Proof. The map

$$h : \{x \in G : |x| = 1\} \times [-\frac{1}{3}, 2] \rightarrow G, \quad (x, p) \mapsto x^{-1} D_{1+p} x$$

is the restriction of a smooth map to a compact set. Since furthermore $h(x, 0) = 0$, we find a constant C such that for all $p \in [-\frac{1}{3}, 2]$ we have

$$|x| = 1 \Rightarrow \|x^{-1} D_{1+p} x\|_2 \leq C |p|.$$

The norm estimate (8) yields

$$|x| = 1 \Rightarrow |x^{-1} D_{1+p} x| \leq C |p|^{\frac{1}{r}}.$$

Setting $p = \frac{t}{t-s} - 1 = \frac{s}{t-s}$, we obtain the estimate

$$|x| = 1 \Rightarrow \left| x^{-1} D_{\frac{t}{t-s}} x \right| \leq C \left| \frac{s}{t-s} \right|^{\frac{1}{r}} \leq C_1 \left| \frac{s}{t} \right|^{\frac{1}{r}},$$

and for arbitrary x with $|x| = r$, we have

$$C_1 \left| \frac{s}{t} \right|^{\frac{1}{r}} \geq \left| (D_r^{-1} x^{-1}) D_{\frac{t}{t-s}} D_r^{-1} x \right| = r^{-1} t \left| (D_t^{-1} x^{-1}) (D_{t-s}^{-1} x) \right|.$$

Finally, multiplying with $r \cdot t^{-1}$ yields

$$C_1 \left| \frac{s}{t} \right|^{\frac{1}{r}} t^{-1} |x| \geq |(D_t^{-1} x^{-1}) (D_{t-s}^{-1} x)| = |(D_{t-s}^{-1} x)^{-1} (D_t^{-1} x)|.$$

\square

Theorem 11. *There are constants k and C and a Schwartz norm $\|\cdot\|_{(N)}$ such that for all $\psi \in \mathcal{S}(G)$, $\delta > 0$, $y \in G$, $\bar{y} \in \tilde{B}_\delta(y)$ it holds that*

$$\int_{G \setminus \tilde{B}_{k\delta}(y)} |K_\psi(x, y) - K_\psi(x, \bar{y})| dx \leq C \|\psi\|_{(N)}. \quad (29)$$

If in addition $\psi(O_t x) = \psi(x)$ for all t and x , then

$$\int_{G \setminus \tilde{B}_{k\delta}(y)} |\overline{K_\psi(y, x)} - \overline{K_\psi(\bar{y}, x)}| dx \leq C \|\psi\|_{(N)}. \quad (30)$$

Proof. Given the assumption about ψ , we have

$$\overline{K(y, x)} = \int_0^\infty \psi_t^*(x \cdot O_t y^{-1}) \frac{dt}{t}.$$

Therefore, (30) can be proven like (29) with minor changes. We prove (29): Because of (20), there is some $k \geq 3$ such that $d(x, y) \geq k\delta \wedge d(y, \bar{y}) < \delta \Rightarrow d(x, \bar{y}) \geq 2\delta$ for every $\delta > 0$. Let $\delta > 0$ and $y, \bar{y} \in G$ such that $d(y, \bar{y}) < \delta$. Let s be a number such that $|O_s y \bar{y}^{-1}| < \delta$ and $|s| \leq \delta$. Lemma 8 yields the estimate

$$\int_{d(x, y) > k\delta} |^{2\delta-s} K_\psi(x, y) - ^{2\delta} K_\psi(x, \bar{y})| \leq C \|\psi\|_{(N_0)}, \quad (31)$$

for some Schwartz norm $\|\cdot\|_{(N_0)}$. Hence, it is sufficient to show that

$$\int_G |K_\psi^{2\delta-s}(x, y) - K_\psi^{2\delta}(x, \bar{y})| dx \leq C \|\psi\|_{(N')}. \quad (32)$$

Observe that by substituting $t - s$ for t , we have

$$K_\psi^{2\delta-s}(x, y) = \int_{2\delta-s}^\infty \psi_t(x \cdot O_{-t} y^{-1}) \frac{dt}{t} = \int_{2\delta}^\infty \frac{\psi_{t-s}(x \cdot O_{-t+s} y^{-1})}{t-s} dt. \quad (33)$$

This transformation yields additional cancellation because afterward, in (32), at the same value of t , the function ψ is evaluated in two points of small homogeneous distance. To improve the estimates, we decompose the kernel integral in (32) further. Setting

$$a(t, x) = \frac{\psi_{t-s}(x \cdot O_{-t+s} y^{-1})}{t-s} \text{ and } d(t, x) = \frac{\psi_t(x \cdot O_{-t} \bar{y}^{-1})}{t},$$

$$b(t, x) = \frac{\psi_{t-s}(x \cdot O_{-t} \bar{y}^{-1})}{t-s} \text{ and } c(t, x) = \frac{\psi \circ D_{t-s}^{-1}(x \cdot O_{-t} \bar{y}^{-1})}{t^{Q+1}},$$

we have

$$\begin{aligned} & \left| K_\psi^{2\delta-s}(x, y) - K_\psi^{2\delta}(x, \bar{y}) \right| \\ & \leq \int_{2\delta}^\infty |a(t, x) - d(t, x)| dt \\ & = \int_{2\delta}^\infty |a(t, x) - b(t, x) + b(t, x) - c(t, x) + c(t, x) - d(t, x)| dt \\ & \leq \int_{2\delta}^\infty |a(t, x) - b(t, x)| dt + \int_{2\delta}^\infty |b(t, x) - c(t, x)| dt + \int_{2\delta}^\infty |c(t, x) - d(t, x)| dt. \end{aligned} \quad (34)$$

Note that in these integrals $t - s \simeq t$. The proof will be completed by showing that the estimate (32) holds with the integrand replaced by each of the three t -integrals at the end of (34). The first of these integrals is

$$\begin{aligned} & \int_G \int_{2\delta}^\infty |a(t, x) - b(t, x)| \, dt \, dx \\ & \leq C \int_{2\delta}^\infty \int_G t^{-Q} |\psi \circ D_{t-s}^{-1}(x \cdot O_{-t+s} y^{-1}) - \psi \circ D_{t-s}^{-1}(x \cdot O_{-t} \bar{y}^{-1})| \, dx \frac{dt}{t}, \end{aligned} \quad (35)$$

which, by Lemma 2, is bounded by

$$C \int_{2\delta}^\infty \int_G t^{-Q} C \|\psi\|_{(N_1)} |D_{t-s}^{-1}[(O_s y) \bar{y}^{-1}]|^\gamma H(x, t) \, dx \frac{dt}{t}, \quad (36)$$

where

$$H(x, t) = \left(\frac{1}{(1 + \|D_{t-s}^{-1}[x \cdot O_{-t+s} y^{-1}]\|_2)^l} + \frac{1}{(1 + \|D_{t-s}^{-1}[x \cdot O_{-t} \bar{y}^{-1}]\|_2)^l} \right)$$

and $l > n$. Note that there is a C such that for every t

$$\int_G H(x, t) \, dx \leq C t^Q.$$

Hence (36) continues as

$$\begin{aligned} & \int_G \int_{2\delta}^\infty |a(t) - b(t)| \, dt \, dx \\ & \leq C \|\psi\|_{(N_1)} \int_{2\delta}^\infty t^{-Q} t^Q |D_{t-s}^{-1}[(O_s y) \bar{y}^{-1}]|^\gamma \frac{dt}{t} \\ & \leq C \|\psi\|_{(N_1)} \int_{2\delta}^\infty t^{-\gamma} \delta^\gamma \frac{dt}{t} \\ & \leq C \|\psi\|_{(N_1)}. \end{aligned} \quad (32ab)$$

The second integral can be written as

$$\begin{aligned} & \int_G \int_{2\delta}^\infty |b(t, x) - c(t, x)| \, dt \, dx \\ & = \int_G \int_{2\delta}^\infty \left| \frac{\psi_{t-s}(x \cdot O_{-t} \bar{y}^{-1})}{t-s} - \frac{\psi \circ D_{t-s}^{-1}(x \cdot O_{-t} \bar{y}^{-1})}{t^{Q+1}} \right| \\ & = \int_{2\delta}^\infty \int_G |(t-s)^{-Q-1} - t^{-Q-1}| \cdot |\psi \circ D_{t-s}^{-1}(x \cdot O_{-t} \bar{y}^{-1})| \, dx \, dt \\ & = \int_{2\delta}^\infty |(t-s)^{-Q-1} - t^{-Q-1}| \int_G |\psi \circ D_{t-s}^{-1}(x)| \, dx \, dt. \end{aligned}$$

This yields the estimate

$$\begin{aligned}
& \int_G \int_{2\delta}^\infty |b(t, x) - c(t, x)| \, dt \, dx \\
& \leq C \int_{2\delta}^\infty t^{\mathcal{Q}-2} |s| \int_G \cdot |\psi \circ D_{t-s}^{-1}(x)| \, dx \, dt \\
& \leq C \int_{2\delta}^\infty t^{\mathcal{Q}-1} \delta \int_G \cdot |\psi \circ D_{t-s}^{-1}(x)| \, dx \, \frac{dt}{t} \\
& \leq C \int_{2\delta}^\infty t^{\mathcal{Q}-1} \delta \cdot (t-s)^{\mathcal{Q}} \|\psi\|_1 \, \frac{dt}{t} \\
& \leq C \|\psi\|_1 \delta \int_{2\delta}^\infty t^{-1} \, \frac{dt}{t} \\
& \leq C \|\psi\|_1 \\
& \leq C \|\psi\|_{(N_2)}.
\end{aligned} \tag{32bc}$$

This was the second part. The third part is

$$\begin{aligned}
& \int_G \int_{2\delta}^\infty |c(t, x) - d(t, x)| \, dt \, dx \\
& = \int_G \int_{2\delta}^\infty \left| \frac{\psi \circ D_{t-s}^{-1}(x \cdot O_{-t}\bar{y}^{-1})}{t^{\mathcal{Q}+1}} - \frac{\psi_t(x \cdot O_{-t}\bar{y}^{-1})}{t} \right| \, dt \, dx \\
& = \int_{2\delta}^\infty \int_G t^{-\mathcal{Q}} |\psi \circ D_{t-s}^{-1}(x \cdot O_{-t}\bar{y}^{-1}) - \psi \circ D_t^{-1}(x \cdot O_{-t}\bar{y}^{-1})| \, dx \, \frac{dt}{t} \\
& = \int_{2\delta}^\infty \int_G t^{-\mathcal{Q}} |\psi(D_{t-s}^{-1}x) - \psi(D_t^{-1}x)| \, dx \, \frac{dt}{t} \\
& \leq C \|\psi\|_{(N)} \int_{2\delta}^\infty \int_G t^{-\mathcal{Q}} |(D_{t-s}^{-1}x)^{-1}(D_t^{-1}x)|^\gamma H(x, t) \, dx \, \frac{dt}{t}.
\end{aligned} \tag{37}$$

The last step relies on Lemma 2, where l has been chosen such that $\int_G |D_t^{-1}x|^\gamma \cdot H(x, t) \, dx < \infty$, and H is the function

$$H(x, t) = \frac{1}{(1 + \|D_{t-s}^{-1}x\|_2)^l} + \frac{1}{(1 + \|D_t^{-1}x\|_2)^l}. \tag{38}$$

Since $|s| \leq \delta$ and $t \geq 2\delta$, we have $\left\|D_{\frac{t}{t-s}}\right\| \leq C$ and $\left\|D_{\frac{t-s}{t}}\right\| \leq C$, that is, $\|D_t^{-1}x\|_2 \simeq \|D_{t-s}^{-1}x\|_2$, and the two summands in (38) are bounded by a constant multiple of the second one. Furthermore,

$$\int_G |D_t^{-1}x|^\gamma \cdot H(x, t) \, dx \leq Ct^{\mathcal{Q}}.$$

Using Lemma 10, we continue estimate (37) with

$$\begin{aligned}
& \int_G \int_{2\delta}^{\infty} |c(t, x) - d(t, x)| \, dt \, dx \\
& \leq C \|\psi\|_{(N_3)} \int_{2\delta}^{\infty} \int_G t^{-Q} \left(C_1 |D_t^{-1} x| \cdot \left(\frac{|s|}{t} \right)^{\frac{1}{p}} \right)^{\gamma} H(x, t) \, dx \, \frac{dt}{t} \\
& \leq C \|\psi\|_{(N_3)} \int_{2\delta}^{\infty} t^{-Q-\frac{\gamma}{p}} \delta^{\frac{\gamma}{p}} \int_G |D_t^{-1} x|^{\gamma} H(x, t) \, dx \, \frac{dt}{t} \\
& \leq C \|\psi\|_{(N_3)} \int_{2\delta}^{\infty} t^{-Q-\frac{\gamma}{p}} \delta^{\frac{\gamma}{p}} t^Q \, \frac{dt}{t} \\
& \leq C \|\psi\|_{(N_3)} \delta^{\frac{\gamma}{p}} \int_{2\delta}^{\infty} t^{-\frac{\gamma}{p}} \, \frac{dt}{t} \\
& \leq C \|\psi\|_{(N_3)}.
\end{aligned} \tag{32cd}$$

Adding (32ab), (32bc), (32cd) finishes the proof of (32) with $N' = \max\{N_1, N_2, N_3\}$. \square

6 Proof of Theorem 1

Proof. The weak type result (3) and the L^p -result (4) for $1 < p < 2$ follow with the help of [10, Theorem 3, p. 19], which relies on a Calderón-Zygmund decomposition of f , here with respect to the quasi-balls \tilde{B} . The relevant premises have been verified in Theorems 4, 5, and 11.

If $2 < p < \infty$, let $q = \frac{p}{p-1}$ and $M = \left\{ g \in \mathcal{S}(G) : \|g\|_q < 1 \right\}$. Note that $q < 2$. With the same arguments as in the proof of boundedness for T , but replacing $\overline{K(y, x)}$ for $K(x, y)$, it can be shown that $\|T^*g\|_q \leq C\|g\|_q$. It follows that

$$\|Tf\|_p = \sup_{g \in M} \langle Tf | g \rangle = \sup_{g \in M} \langle f | T^*g \rangle \leq \sup_{g \in M} \|f\|_p \|T^*g\|_q \leq C_p \|f\|_p$$

for any $f \in L^2 \cap L^p(G)$. \square

7 Example

Let G_1 be the real vector space $\mathbb{C} \times \mathbb{R}^{n-2}$. We denote its elements by

$$x = (u, x'), \quad u \in \mathbb{C}, \quad x' = (x_3, \dots, x_n) \in \mathbb{R}^{n-2}.$$

Let $a, a_3, \dots, a_n > 0$. We introduce a family of dilations with $Q = 2a + \sum_{j=3}^n a_j$ by

$$D_r x = (r^a u, r^{a_3} x_3, \dots, r^{a_n} x_n)$$

and a family of rotations by $O_t(u, x') = (e^{it}u, x')$. There are several ways to define homogeneous norms on G_1 , each of which has its own advantages. It is known that on any homogeneous group any two homogeneous norms $|\cdot|$ and $|\cdot|'$ are equivalent in the

sense that $|\cdot| \simeq |\cdot|'$. The volumes of the corresponding balls and quasi-balls are also equivalent in the sense that $\mu(\tilde{B}_r \cdot z) \simeq \mu(B'_r \cdot z)$ and $\mu(\tilde{B}_r(z)) \simeq \mu(\tilde{B}'_r(z))$ if the balls B, \tilde{B} and B', \tilde{B}' are defined in terms of the homogeneous norms $|\cdot|$ and $|\cdot|'$ respectively. Let us consider the homogeneous norms

$$\begin{aligned} |x| &= \max\{|u|_{\mathbb{C}}^{\frac{1}{a}}, |x_3|_{\mathbb{R}}^{\frac{1}{a_3}}, \dots, |x_n|_{\mathbb{R}}^{\frac{1}{a_n}}\}, \\ |x|' &= \inf\{r > 0 : \|D_{r^{-1}}(x)\|_2 < 1\}, \\ |x|'' &= \left(|u|_{\mathbb{C}}^{\frac{2}{a}} + \sum_{j=3}^n |x_j|_{\mathbb{R}}^{\frac{2}{a_j}}\right)^{\frac{1}{2}}. \end{aligned} \quad (39)$$

The second one has the advantage of being smooth away from the origin, while the first one allows effortless calculations. Throughout this section, we will use $|\cdot|$, and theorems will be valid for any homogeneous norm.

If $y = (v, y') \in G_1$, and if we use the homogeneous norm given by (39), the quasi-metric (19) has the form

$$\begin{aligned} d(x, y) &= \inf\{r > 0 : \exists s : |s| \leq r \wedge |x - O_s y| < r\}, \\ &= \min_{s \in \mathbb{R}} \max\left\{|s|, |u - e^{is}v|_{\mathbb{C}}^{\frac{1}{a}}, |x_3 - y_3|_{\mathbb{R}}^{\frac{1}{a_3}}, |x_n - y_n|_{\mathbb{R}}^{\frac{1}{a_n}}\right\}. \end{aligned}$$

With y and r fixed, we have

$$\begin{aligned} x &\in \tilde{B}_r(y) \\ \Leftrightarrow d(x, y) &< r \\ \Leftrightarrow \max\left\{\min_{|s| \leq r} |u - e^{is}v|_{\mathbb{C}}^{\frac{1}{a}}, |x_3 - y_3|_{\mathbb{R}}^{\frac{1}{a_3}}, |x_n - y_n|_{\mathbb{R}}^{\frac{1}{a_n}}\right\} &< r. \end{aligned} \quad (40)$$

Lemma 12. *For all $y = (v, y')$ in G_1 and all $r > 0$, the volume $\mu(\tilde{B}_r(y))$ is bounded from above and from below by a constant multiple of*

$$r^Q + r^{Q-a} |v| \min\{1, r\}.$$

Proof. As a first step, we consider the case $n = 2$. For $v \in \mathbb{C}$, $\varphi \in \mathbb{R}$, and $r > 0$ let

$$B'(s, \varphi, v) = \bigcup_{\alpha \in [-\varphi, \varphi]} \{z : \|z\|_2 < s\} + e^{i\alpha}v.$$

By some calculations corresponding to Fig. 2, we see that

$$\begin{aligned} \mu(B'(s, \varphi, v)) &\leq \pi s^2 + \pi(|v| + s)^2 \frac{\min\{2\pi, \varphi\}}{2\pi} - \pi(|v| - s)^2 \frac{\min\{2\pi, \varphi\}}{2\pi} \\ &= \pi s^2 + 2s \cdot |v| \min\{2\pi, \varphi\} \\ &\leq 2\mu(B(s, \varphi, v)). \end{aligned} \quad (41)$$

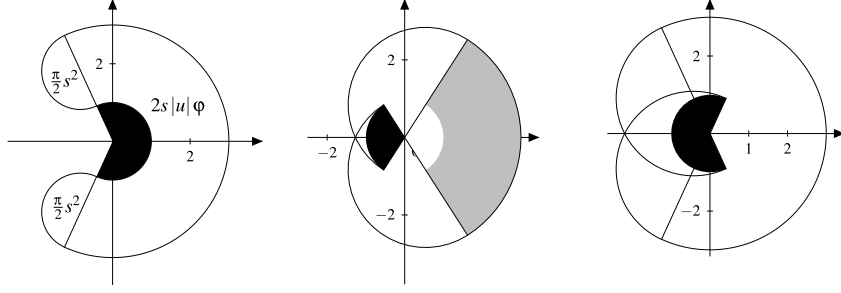


Figure 2: The sets $B'(1, 2, 2)$, $B'(2, 1, 1)$ and $B'(2, 2, 1)$. The size of the black areas is $\pi(|u| - s)^2 \frac{\min\{2\pi, \varphi\}}{2\pi}$, see equation (41).

Equality holds in the first two lines of (41) if $s < |v|$ and $\varphi < \frac{\pi}{2}$. Setting $s = r^a$ and $\varphi = r$, (41) yields

$$\mu(\tilde{B}_r(y)) = \mu(B'(r^a, r, v)) \simeq \pi r^{2a} + 2r^a \cdot |v| \min\{2\pi, r\}, \quad (42)$$

so we are done with the case $n = 2$. Now let $n \geq 3$. Since

$$\tilde{B}_r(y) = B'(r^a, r, y) \times \prod_{j=3}^n (y_j - r^{aj}, y_j + r^{aj})$$

we have

$$\begin{aligned} \mu(\tilde{B}_r(y)) &= \mu(B'(r^a, r, y)) \cdot (2r)^{Q-2a} \\ &\simeq (r^{2a} + r^a \cdot |v| \min\{1, r\}) r^{Q-2a}, \end{aligned}$$

which completes the proof. \square

The following Lemma roughly says that in G_1 , the volume of the balls \tilde{B} grows at least as in spaces of dimension $Q - a$.

Lemma 13. *There is a constant C such that for any $y \in G_1$, $r \geq 0$ and $j \in \mathbb{N}$ we have*

$$\mu(\tilde{B}_{2^j r}(y)) \geq C 2^{j(Q-a)} \mu(\tilde{B}_r(y)).$$

Proof. \square

Using Lemma 12 we obtain the estimate

$$\begin{aligned} \mu(\tilde{B}_{2^j r}(y)) &\simeq (2^j r)^Q + (2^j r)^{Q-a} |v| \min\{1, 2^j r\} \\ &= 2^{j(Q-a)} (2^{ja} r^Q + r^{Q-a} |v| \min\{1, 2^j r\}) \\ &\geq 2^{j(Q-a)} (r^Q + r^{Q-a} |v| \min\{1, r\}) \\ &\simeq C 2^{j(Q-a)} \mu(\tilde{B}_r(y)). \end{aligned} \quad (43)$$

Lemma 14. *There is a constant C_1 such that for all $R \geq 1, r > 0$ and $x, y \in G_1$ we have the estimate*

$$\int_r^{2r} 1_{\{t : |x - O_{-t}y| < Rt\}} \frac{dt}{t} \leq C_1 \frac{R^a r^Q}{\mu(\tilde{B}_r(y))}. \quad (44)$$

Furthermore, for any $p > a - Q$ there is a constant C_2 such that

$$\int_r^\infty t^{-Q-p} \cdot 1_{\{t : |x - O_{-t}y| < Rt\}} \frac{dt}{t} \leq C_2 \frac{R^a r^{-p}}{\mu(\tilde{B}_r(y))} \quad (45)$$

for all $r > 0$ and all $x, y \in G_1$. These statements are also true when O_t is substituted for O_{-t} or when x and y are exchanged on one side.

Proof. For some constants C_1, C_2 we have $\mu(\tilde{B}_r(y)) \leq C_1 r^Q + C_2 r^{Q-a} |v| \min\{1, r\}$ by Lemma 12. If $\mu(\tilde{B}_r(y)) < 2C_1 r^Q$, then we are done with (44) because the left side of (44) is bounded by a constant. Otherwise we have $\mu(\tilde{B}_r(y)) \geq 2C_1 r^Q$ and

$$\mu(\tilde{B}_r(y)) \leq 2C_2 r^{Q-a} |v| \min\{1, r\},$$

where C_2 does not depend on y or r . Note that $v \neq 0$. Assume for the moment that

$$\int_r^{2r} 1_{\{t : |x - O_{-t}y| < Rr\}} dt \lesssim \frac{R^a r^{a+1}}{|v| \min\{r, 1\}}. \quad (46)$$

Then, with $2R$ substituted for R , we get

$$\begin{aligned} \int_r^{2r} 1_{\{t : |x - O_{-t}y| < Rt\}} \frac{dt}{t} &\leq r^{-1} \int_r^{2r} 1_{\{t : |x - O_{-t}y| < R2r\}} dt \\ &\lesssim \frac{(2R)^a r^Q}{r^{Q-a} |v| \min\{r, 1\}} \\ &\lesssim \frac{R^a r^Q}{\mu(\tilde{B}_r(y))}, \end{aligned}$$

and the proof of (44) would be finished. So let us prove assumption (46). Note that by (39) we have

$$|x - O_{-t}y| < Rr \quad \Rightarrow \quad |u - e^{-it}v|_{\mathbb{C}}^{\frac{1}{a}} < Rr.$$

For any $0 < r < r' \leq r + 2\pi$, it follows with the help of Fig. 3 that

$$\begin{aligned} \int_r^{r'} 1_{\{t : |x - O_{-t}y| < Rr\}} dt &\leq \mu \left\{ t \in [0, 2\pi] : |u - e^{-it}v| < (Rr)^a \right\} \\ &= \mu \left\{ t \in [0, 2\pi] : \left| \frac{u}{|v|} - e^{-it} \frac{v}{|v|} \right| < \frac{(Rr)^a}{|v|} \right\} \\ &\leq 2\pi \frac{(Rr)^a}{|v|} \\ &\lesssim \frac{R^a r^a}{|v|}. \end{aligned} \quad (47)$$

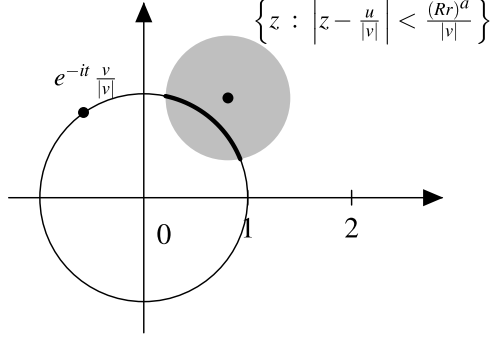


Figure 3: The length of the thick line is smaller than the circumference of the shaded area, see (47).

If $r < 1$, then (46) follows from (47) with $r' = 2r$ because their right hand sides are equal. Otherwise, when $r \geq 1$, we decompose the integral into the sum

$$\begin{aligned} \int_r^{2r} 1_{\{t : |x - O_{-t}y| < Rr\}} dt &\leq \sum_{k=0}^{\lfloor r \rfloor} \int_{r+k}^{r+k+1} 1_{\{t : |x - O_{-t}y| < Rr\}} dt \\ &\lesssim \sum_{k=0}^{\lfloor r \rfloor} \frac{R^a (r+k)^a}{|v|} \\ &\lesssim \frac{R^a r^{a+1}}{|v|}, \end{aligned}$$

which proves (46) and finishes the proof of (44). A dyadic decomposition of the interval $]0, \infty[$, equation (44) and Lemma 13 now yield

$$\begin{aligned} \int_r^\infty t^{-Q-p} \cdot 1_{\{t : |x - O_{-t}y| < Rt\}} \frac{dt}{t} &= \sum_{j=0}^\infty \int_{2^j r}^{2^{j+1} r} t^{-Q-p} \cdot 1_{\{t : |x - O_{-t}y| < Rt\}} \frac{dt}{t} \\ &\leq \sum_{j=0}^\infty (2^j r)^{-Q-p} \cdot C_1 \cdot \frac{R^a (2^j r)^Q}{\mu(\tilde{B}_{2^j r}(y))} \\ &= C_1 R^a r^{-p} \sum_{j=0}^\infty \frac{2^{-jp}}{\mu(\tilde{B}_{2^j r}(y))} \frac{\mu(\tilde{B}_r(y))}{\mu(\tilde{B}_r(y))} \\ &\lesssim \frac{R^a r^{-p}}{\mu(\tilde{B}_r(y))} \sum_{j=0}^\infty 2^{-jp} \cdot 2^{j(a-Q)} \\ &\lesssim \frac{R^a r^{-p}}{\mu(\tilde{B}_r(y))}, \end{aligned}$$

which proves (45). □

8 Calderón-Zygmund kernels

There is a number of related notions of "standard" Calderón-Zygmund kernels in the context of homogeneous spaces, see [10, p. 29][3][9]. We say that K is a Calderón-Zygmund kernel with respect to a quasi-metric d , corresponding quasi-balls \tilde{B} and a measure μ , if there exist constants $k > 1$, $\varepsilon > 0$, and C such that for all x, y, \bar{y} the following estimates hold:

$$|K(x, y)| \leq \frac{C}{\mu(\tilde{B}_{d(x, y)}(y))} \quad \text{and} \quad (48)$$

$$\begin{aligned} & |K(y, x) - K(\bar{y}, x)| + |K(x, y) - K(x, \bar{y})| \\ & \leq C \left(\frac{d(\bar{y}, y)}{d(x, y)} \right)^\varepsilon \cdot \frac{1}{\mu(\tilde{B}_{d(x, y)}(y))} \quad \text{if } k \cdot d(y, \bar{y}) < d(x, y). \end{aligned} \quad (49)$$

There are hints that the kernels introduced in Sect. 5 are of this type. We examine an example in the setting made up in Sect. 7.

Theorem 15. *Let $G = G_1$, D , d , and O as in Sect. 7 and let K be a kernel K_ψ , where $\psi(O_t x) = \psi(x)$, $\text{supp } \psi \subseteq B_1$ and $\int \psi = 0$, as defined in Sect. 5. Then there exist k , c , ε , and C such that K is a Calderón-Zygmund kernel in the sense of (48) and (49).*

Proof. We choose some $k \geq 4$ such that under the condition $k \cdot d(y, \bar{y}) < d(x, y)$, we have $d(x, y) \simeq d(x, \bar{y})$ as in (20) and furthermore $4d(y, \bar{y}) < d(x, \bar{y})$. Then it is sufficient to show (48) and (49) with $d(x, y)$ replaced by $\Delta = \min\{d(x, y), d(x, \bar{y})\}$. Note that if $x - O_{-t}y \in \text{supp } \psi_t$ or $O_t x - y \in \text{supp } \psi_t$, then $|x - O_{-t}y| < t$ and consequently $d(x, y) < t$. Thus we have

$$K(x, y) = \int_{\Delta}^{\infty} \psi_t(x - O_{-t}y) \frac{dt}{t}$$

and Lemma 14 with $p = 0$ yields

$$\begin{aligned} |K(x, y)| & \leq \left| \int_{\Delta}^{\infty} t^{-Q} \|\psi\|_{\infty} \cdot 1_{\{t : |x - O_{-t}y| < t\}} \frac{dt}{t} \right| \\ & \leq C \|\psi\|_{\infty} \frac{1}{\mu(\tilde{B}_{\Delta}(y))}, \end{aligned}$$

proving (48). We continue with the proof of (49), which is trivial if $y = \bar{y}$, so we assume $y \neq \bar{y}$. Due to the rotational symmetry of ψ , we have

$$K(\bar{y}, x) = \int_{\Delta}^{\infty} \psi_t(O_t \bar{y} - x) \frac{dt}{t}. \quad (50)$$

Setting $\delta = d(y, \bar{y})$, the estimate $0 < 4\delta < \Delta$ holds, and it suffices to verify that

$$|K(x, y) - K(x, \bar{y})| \leq C \left(\frac{\delta}{\Delta} \right)^\varepsilon \cdot \frac{1}{\mu(\tilde{B}_{\Delta}(y))} \quad (51)$$

and

$$|K(y, x) - K(\bar{y}, x)| \leq C \left(\frac{\delta}{\Delta} \right)^\varepsilon \cdot \frac{1}{\mu(\tilde{B}_{\Delta}(y))}. \quad (52)$$

Because of (50), (52) can be shown with the same techniques as (51), so we prove (51) only. Choose some real number s , $|s| \leq \delta$, such that $|O_s y - \bar{y}| \leq \delta$. Then $|s| \leq \frac{1}{4}\Delta$ and

$$\begin{aligned}
& |K(x, y) - K(x, \bar{y})| \\
&= \left| \int_{\Delta}^{\infty} t^{-Q-1} [\psi(D_t^{-1}(x - O_{-t}y)) - \psi(D_t^{-1}(x - O_{-t}\bar{y}))] dt \right| \\
&= \left| \int_{\frac{3}{4}\Delta}^{\infty} [(t-s)^{-Q-1} \psi(D_{t-s}^{-1}(x - O_{-t+s}y)) - t^{-Q-1} \psi(D_t^{-1}(x - O_{-t}\bar{y}))] dt \right| \\
&\leq \int_{\frac{3}{4}\Delta}^{\infty} (t-s)^{-Q-1} \cdot |\psi(D_{t-s}^{-1}(x - O_{-t+s}y)) - \psi(D_{t-s}^{-1}(x - O_{-t}\bar{y}))| dt \\
&\quad + \int_{\frac{3}{4}\Delta}^{\infty} |(t-s)^{-Q-1} - t^{-Q-1}| \cdot |\psi(D_{t-s}^{-1}(x - O_{-t}\bar{y}))| dt \\
&\quad + \int_{\frac{3}{4}\Delta}^{\infty} t^{-Q-1} \cdot |\psi(D_{t-s}^{-1}(x - O_{-t}\bar{y})) - \psi(D_t^{-1}(x - O_{-t}\bar{y}))| dt \\
&= \int_{\frac{3}{4}\Delta}^{\infty} f_1(t) \frac{dt}{t} + \int_{\frac{3}{4}\Delta}^{\infty} f_2(t) \frac{dt}{t} + \int_{\frac{3}{4}\Delta}^{\infty} f_3(t) \frac{dt}{t},
\end{aligned}$$

where f_1 , f_2 , and f_3 are the integrands in the preceding lines. They are supported in the sets

$$\begin{aligned}
& \text{supp } f_1 \subseteq \{t > 0 : |x - O_{-t}\bar{y}| < 2t \text{ or } |x - O_{-t}(O_s y)| < 2t\}, \\
& \text{supp } f_2, \text{supp } f_3 \subseteq \{t > 0 : |x - O_{-t}\bar{y}| < 2t\}.
\end{aligned}$$

Lemma 14 yields

$$\int_{\frac{3}{4}\Delta}^{\infty} t^{-Q-p} \cdot 1_{\{\text{supp } f_j\}} \frac{dt}{t} \lesssim \frac{\Delta^{-p}}{\mu(\tilde{B}_{\Delta}(x))} \lesssim \frac{\Delta^{-p}}{\mu(\tilde{B}_{\Delta}(y))}, \quad j = 1, 2, 3. \quad (53)$$

We now complete the proof by estimating the integrands f_j against appropriate powers of t . We have

$$\begin{aligned}
f_1(t) &= (t-s)^{-Q} \cdot |\psi(D_{t-s}^{-1}(x - O_{-t+s}y)) - \psi(D_{t-s}^{-1}(x - O_{-t}\bar{y}))| \\
&\lesssim \|\nabla \psi\|_{\infty} t^{-Q} \|D_{t-s}^{-1}(O_s y - \bar{y})\|_2 \\
&= \|\nabla \psi\|_{\infty} t^{-Q} \left\| D_{\frac{\delta}{t-s}} D_{\delta}^{-1}(O_s y - \bar{y}) \right\|_2 \\
&\leq \|\nabla \psi\|_{\infty} t^{-Q} \left\| D_{\frac{\delta}{t-s}} \right\| \\
&\leq C_1 \|\nabla \psi\|_{\infty} \delta^{\gamma} t^{-Q-\gamma},
\end{aligned}$$

where $\gamma = \min\{a, a_2, \dots, a_n\}$. Together with (53) this implies the inequality

$$\begin{aligned}
& \int_{\frac{3}{4}\Delta}^{\infty} f_1(t) \frac{dt}{t} \\
&\leq C_1 \|\nabla \psi\|_{\infty} \delta^{\gamma} \int_{\frac{3}{4}\Delta}^{\infty} t^{-Q-\gamma} \cdot 1_{\{\text{supp } f_1\}} \frac{dt}{t} \\
&\leq C_2 \|\nabla \psi\|_{\infty} \left(\frac{\delta}{\Delta}\right)^{\gamma} \frac{1}{\mu(\tilde{B}_{\Delta}(X))}.
\end{aligned} \quad (54)$$

In a similar way we obtain

$$\begin{aligned}
f_2(t) &= |(t-s)^{-Q-1} - t^{-Q-1}| \cdot |\psi(D_{t-s}^{-1}(x - O_{-t}\bar{y}))| t \\
&\leq C \|\psi\|_\infty \delta t^{-Q-1}, \\
\int_{\frac{3}{4}\Delta}^\infty f_2(t) \frac{dt}{t} &\leq \|\nabla \psi\|_\infty r \int_{\frac{3}{4}\Delta}^\infty t^{-Q-1} \cdot 1_{\{\text{supp } f_2\}} \frac{dt}{t} \leq \|\psi\|_\infty \frac{\delta}{\Delta} \cdot \frac{1}{\mu(\tilde{B}_\Delta(y))}. \quad (55)
\end{aligned}$$

Furthermore, we have the estimate

$$\begin{aligned}
f_3(t) &= t^{-Q} \cdot |\psi(D_{t-s}^{-1}(x - O_{-t}\bar{y})) - \psi(D_t^{-1}(x - O_{-t}\bar{y}))| \\
&\leq \|\nabla \psi\|_\infty t^{-Q} \cdot \|(D_{t-s}^{-1} - D_t^{-1})(x - O_{-t}\bar{y})\|_2 \\
&= \|\nabla \psi\|_\infty t^{-Q} \cdot \|(D_{1-\frac{s}{t}}^{-1} - \text{Id})D_t^{-1}(x - O_{-t}\bar{y})\|_2 \\
&\leq \|\nabla \psi\|_\infty t^{-Q} \cdot \|(D_{1-\frac{s}{t}}^{-1} - \text{Id})\|.
\end{aligned}$$

Note that $p \mapsto D_{1-p}^{-1} - \text{Id}$ is smooth, mapping 0 to 0, and that $|\frac{s}{t}| \leq |\frac{\delta}{t}| \leq \frac{1}{3}$. So we find a constant such that

$$f_3(t) \leq C \|\nabla \psi\|_\infty \delta t^{-Q-1},$$

and

$$\int_{\frac{3}{4}\Delta}^\infty f_3(t) \frac{dt}{t} \leq \|\nabla \psi\|_\infty \delta \int_{\frac{3}{4}\Delta}^\infty t^{-Q-1} \cdot 1_{\{\text{supp } f_3\}} \frac{dt}{t} \leq \|\nabla \psi\|_\infty \frac{\delta}{\Delta} \cdot \frac{1}{\mu(\tilde{B}_\Delta(y))}. \quad (56)$$

Adding inequalities (54) - (56) and using $\frac{\delta}{\Delta} \leq 1$, we obtain a constant C such that

$$|K(x, y) - K(x, \bar{y})| \leq C \cdot (\|\psi\|_\infty + \|\nabla \psi\|_\infty) \left(\frac{\delta}{\Delta}\right)^{\min\{1, \gamma\}} \cdot \frac{1}{\mu(\tilde{B}_\Delta(y))}.$$

□

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References

- [1] H. F. Bloch: Ein Integraloperator auf homogenen Gruppen. (unpublished)
- [2] R. R. Coifman, G. Weiss: Analyse Harmonique Non-Commutative sur Certains Espaces Homogenes. Lecture Notes in Mathematics, Vol. 242 (1971)
- [3] D. Deng, Y. Han and Y. Meyer: Harmonic Analysis on Spaces of Homogeneous Type. Lecture Notes in Mathematics, Springer (2009)

- [4] R. Farwig, T. Hishida, D. Müller: L^q -Theory of a Singular “Winding” Integral Operator Arising from Fluid Dynamics. *Pacific Journal of Mathematics*, Vol. 215, p. 306 (2004)
- [5] G. B. Folland: *Harmonic Analysis in Phase Space*. Princeton University Press (1989)
- [6] G. B. Folland, E. M. Stein: *Hardy Spaces on Homogeneous Groups*. Princeton University Press (1982)
- [7] L. Grafakos: *Classical Fourier Analysis*, 2nd ed. Springer (2008)
- [8] J. Hilgert, K.-H. Neeb: *Lie-Gruppen und Lie-Algebren*, Vieweg, p. 277 (1991)
- [9] S. Hofmann: Weighted Norm Inequalities and Vector Valued Inequalities for Certain Rough Operators. *Indiana University Mathematics Journal*, Vol. 42, No.1, pp. 1-14 (1993)
- [10] E. M. Stein: *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*. Princeton University Press (1993)